Nonexistence of triples of nonisomorphic connected graphs with isomorphic connected P_3-graphs *

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Abstract

In the paper "Broersma and Hoede, *Path graphs*, J. Graph Theory **13** (1989) 427-444", the authors proposed a problem whether there is a triple of mutually nonisomorphic connected graphs which have an isomorphic connected P_3-graph. For a long time, this problem remains unanswered. In this paper, we give it a negative answer that there is no such triple, and thus completely solve this problem.

Keywords: path graph, connected, isomorphism

1 Introduction

Broersma and Hoede [3] generalized the concept of line graphs to that of path graphs by defining adjacency as follows. Let k be a positive integer, and P_k and C_k denote a path and a cycle with k vertices, respectively. Let $\pi_k(G)$ be the set of all P_k's in G. The *path graph* $P_k(G)$ of G is a graph with vertex set $\pi_k(G)$ in which two P_k's are adjacent whenever their union is a path P_{k+1} or a cycle C_k. Broersma and Hoede got many results on P_3-graphs, especially, described two infinite classes of pairs of nonisomorphic connected graphs which have isomorphic connected P_3-graphs. They also raised a number of unsolved problems or questions, all of which have been solved during

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these year, but only the following one remains unanswered.

Problem. Whether there exists a triple of mutually nonisomorphic connected graphs which have an isomorphic connected P_3-graph ?

For $k = 2$, i.e., line graphs, from Whitney's result (see [4]) it is not difficult to see that the problem has a negative answer. In [5] the authors showed that for $k \geq 4$ there are not only triples of but also arbitrarily many mutually nonisomorphic connected graphs with isomorphic connected P_k-graphs. However, interestingly we will show in this paper that for $k = 3$ there does not exist any triple of mutually nonisomorphic connected graphs with an isomorphic connected P_3-graph, just like the case for $k = 2$ but very different from the case for $k \geq 4$. Note that If one drops the connectedness of the original graph or its P_3-graph, then it is easy to find arbitrarily many mutually nonisomorphic graphs with an isomorphic P_3-graph.

2 Preliminaries

All graphs in this paper are undirected, finite and simple. We follow the terminology and notations used in [1, 2]. If σ is an isomorphism from G to H, then σ induces a P_k-isomorphism σ^* from G to H, where $\sigma^*(a_1a_2\cdots a_k) = \sigma(a_1)\sigma(a_2)\cdots\sigma(a_k)$ for all $a_1a_2\cdots a_k \in \pi_k(G)$. A P_k-isomorphism τ is *induced* if $\tau = \sigma^*$ for some isomorphism σ. If τ_i is a P_k-isomorphism from G_i to H_i for $i = 1$ and 2, then we say that τ_1 and τ_2 are *equivalent* if there are isomorphisms σ and ρ from G_1 to G_2 and H_1 to H_2, respectively, such that $\tau_1 = (\rho^*)^{-1} \circ \tau_2 \circ \sigma^*$.

Define an *i-thorn* to be a P_3 with exactly i ($i = 1$ or 2) terminal ends in G. Let $T_i(G)$ be the set of i-thorns in G. We say that two P_3-isomorphisms τ_i from G_i to H_i for $i = 1$ and 2, are *T-related* if *(i)* G_1 and G_2 differ only in their star components, so do H_1 and H_2; *(ii)* $|T_2(G_1)| = |T_2(G_2)|$; and *(iii)* $\tau_1(\alpha) = \tau_2(\alpha)$ for every $\alpha \in \pi_3(G_1) - T_2(G_1) = \pi_3(G_2) - T_2(G_2)$.

Consider two 1-thorns abc and abd where $deg(a) \geq 2$ and $deg(c) = deg(d) = 1$, then swapping abc and abd gives a P_3-isomorphism, which we call a *B-swap*.

Suppose $abcde$ is a P_5 in G such that both abc and cde are terminal 1-thorns, i.e., $deg(a) = deg(e) = 1$ and $deg(c) = 2$, then swapping abc and cde gives a P_3-isomorphism, which we call an *S-swap*.

For distinct $a, b \in V(G)$, let $D_{a,b}$ denote the subgraph of G consisting of the union of all P_3's with ends a and b and with middle vertex of degree 2 in G. If $D_{a,b}$ is nonempty we call it a *diamond* with ends a and b. We usually

write $V(D_{a,b}) - \{a,b\}$ as $\{c_1, c_2, \cdots, c_k\}$ and call k the *width* of $D_{a,b}$, and refer to $D_{a,b}$ as a k-diamond. Note that if $a \sim b$, the edge ab is not included in $D_{a,b}$. To distinguish the two possibilities, we say that the diamond $D_{a,b}$ is *braced* if $a \sim b$ and *unbraced* otherwise. For $1 \leq i < j \leq k$, the P_3's ac_ib are called *diamond paths* while the pair of P_3's c_iac_j and c_ibc_j is called a *diamond pair*. Then swapping c_iac_j and c_ibc_j gives a P_3-isomorphism, which we call a *D-swap*.

Suppose τ_1 and τ_2 are P_3-isomorphisms from G to H. We say that τ_1 and τ_2 are *B-related* if $\tau_2^{-1} \circ \tau_1$ is the identity or a composition of B-swaps. The *S-related* and *D-related* are defined similarly. We use joins of these four equivalence relations: for example, two P_3-isomorphisms are *$TBSD$-related* if we can get from one to the other by a chain of zero or more T-, B-, S- and/or D-relations.

The following is the main result of [1], based on which we shall solve our problem by case analysis.

Theorem 2.1 *Let τ be a P_3-isomorphism from G to H such that at least one of G or H is connected. Then τ is one of the following:*

(i) T-related to a P_3-isomorphism of generalized $K_{3,3}$ type;
(ii) of special Whitney type;
(iii) D-related to a P_3-isomorphism of Whitney type$3, 4, 5$ or 6;
(iv) D-related to a P_3-isomorphism of bipartite type; or
(v) $TBSD$-related to an induced P_3-isomorphism.

The definition for each of the above types will be given in the successive subsections.

For solving our problem, in Theorem 2.1 we only need to consider that the original graphs G and H are nonisomorphic connected graphs with $T_2(G) = T_2(H) = \emptyset$. Below, we will analyze the types in Theorem 2.1 case by case in details.

2.1 Generalized $K_{3,3}$ type

First, we introduce the following notation which is used in the definition of generalized $K_{3,3}$ type. We write $(c,d)ab(e,f) \mapsto uvwxu$ if G contains the edges ab, ac, ad, be, bf, H contains the C_4 $uvwxu$, and τ maps $cab \mapsto xuv$, $dab \mapsto vwx$, $abe \mapsto uvw$ and $abf \mapsto wxu$. We also write $abc(d,e) \mapsto uvwxy$ if G contains the edges ab, bc, cd, ce, H contains the P_5 $uvwxy$, and τ maps $abc \mapsto vwx$, $bcd \mapsto uvw$ and $bce \mapsto wxy$. This notation will be reversed (e.g.,

$abcda \mapsto (w,x)uv(y,z))$ as needed. Then, define the generalized $K_{3,3}$ type as follows:

Either τ or τ^{-1} as in the following cases (i) through (vii), or any equivalent P_3-isomorphism, is said to be of *generalized* $K_{3,3}$ *type*.

(i) $(c,d)ab(e,f) \mapsto u_1v_1u_2v_2u_1$, and cad and ebf map to P_3 components of H.

(ii) $(c,d)ab(e,f) \mapsto u_1v_1u_2v_2u_1$, $kebfh \mapsto yv_3u_1(v_1,v_2)$, and cad maps to a P_3 component.

(iii) $(c,d)ab(e,f) \mapsto u_1v_1u_2v_2u_1$, $(k,l)eb(a,f) \mapsto u_1v_1u_2v_3u_1$, $(h,i)fb(a,e) \mapsto u_1v_2u_2v_3u_1$, and cad, kel and hfi map to P_3 components.

(iv) $(c,d)ab(e,f) \mapsto u_1v_1u_2v_2u_1$, $ecadg \mapsto xu_3v_1(u_1,u_2)$, and $cebfh \mapsto yv_3u_1(v_1,v_2)$. Note that G and H are connected and isomorphic.

(v) $(c,d)ab(e,f) \mapsto u_1v_1u_2v_2u_1$, $ebfhe \mapsto (v_1,v_2)u_1v_3(y,z)$, and cad maps to yv_3z. Again G and H are connected and isomorphic.

(vi) $(c,d)ab(e,f) \mapsto u_1v_1u_2v_2u_1$, $(c,d)eb(a,f) \mapsto u_1v_1u_2v_3u_1$, $(h,i)fb(a,e) \mapsto u_1v_2u_2v_3u_1$, $aceda \mapsto (w,x)u_3v_1(u_1,u_2)$, and hfi maps to wu_3x. Again G and H are connected and isomorphic.

(vii) The construction on $K_{3,3}$; $G \cong H \cong K_{3,3}$.

Remark 1. For generalized $K_{3,3}$ type, it is easy to get the following results:

1. For cases (i), (ii) and (iii), G and H are nonisomorphic, but H is not connected and there are isolated vertices in $P_3(G)$ and $P_3(H)$.
2. For cases (iv) and (vii), G and H are connected with $T_2(G) = T_2(H) = \emptyset$, but G and H are isomorphic.
3. For cases (v) and (vi), G and H are connected, but are isomorphic and there are isolated vertices in $P_3(G)$ and $P_3(H)$.

Thus there is no pair of nonisomorphic connected graphs with isomorphic connected P_3-graphs in generalized $K_{3,3}$ type.

2.2 Special Whitney type

Let SW be the graph obtained by subdividing each edge of $K_{1,3}$ exactly once, then $P_3(SW) \cong C_6$. τ is a P_3-isomorphism from SW to C_6, then we say τ, τ^{-1} or any equivalent P_3-isomorphism is of *special Whitney type*.

It is clear that SW and C_6 are two nonisomorphic connected graphs with isomorphic connected P_3-graphs.

2.3 Whitney type $3, 4, 5$ or 6

In this subsection, we begin with a general idea which will be used here and in the next subsection. Suppose F is a graph. A *diamond inflation* of F is a graph obtained by replacing each edge $ab \in E(F)$ by an unbraced s_{ab}-diamond $D_{a,b}$ ($s_{a,b} \geq 1$), and adding t_a terminal edges incident with each $a \in V(F)$ ($t_a \geq 0$). Suppose φ is an edge-isomorphism between graphs F and $F^{'}$, and suppose I and $I^{'}$ are diamond inflations of F and $F^{'}$, respectively, with the following property: for every $ab \in E(F)$, if $\varphi(ab) = uv$ then (i) $s_{uv} = s_{ab}$ and (ii) $t_u + t_v = t_a + t_b$. Obtain G and H from I and $I^{'}$, respectively, by adding star components to one of them (if necessary) to make the numbers of 2-thorns equal. Then we can define a P_3-isomorphism τ from G to H and say that τ is a *diamond inflation* of φ.

Remark 2. If $D_{a,b}$ is a nontrivial diamond (i.e., $s_{a,b} > 1$) in G, then there exists a unique and nontrivial diamond $D_{u,v}$ in H (see the proof in [1]).

The type in this subsection is related to Whitney's exceptional edge-isomorphisms which is stated as follows:

Theorem 2.2 (Whitney [6]) *Suppose that φ is an edge-isomorphism from G to H where G and H are both connected. If φ is not induced, then $i = |E(G)| = |E(H)| \in \{3, 4, 5, 6\}$, G and H are isomorphic to W_i and $W_i^{'}$ in some order, and φ is equivalent to φ_i or φ_i^{-1}, where*

(i) $W_6 \cong W_6^{'} \cong K_4$, with $V(W_6) = \{a, b, c, d\}$, $V(W_6^{'}) = \{u, v, w, x\}$, and φ_6 maps $ab \mapsto uv$, $ac \mapsto uw$, $ad \mapsto vw$, $bc \mapsto ux$, $bd \mapsto vx$ and $cd \mapsto wx$;

(ii) $W_5 = W_6 - cd$, $W_5^{'} = W_6^{'} - wx$ and $\varphi_5 = \varphi_6|E(W_5)$;

(iii) $W_4 = W_6 - \{bd, cd\}$, $W_4^{'} = W_6^{'} - \{vx, wx\}$ and $\varphi_4 = \varphi_6|E(W_4)$; and

(iv) $W_3 = W_6 - \{bc, bd, cd\} \cong K_{1,3}$, $W_3^{'} = W_6^{'} - x \cong K_3$, and $\varphi_3 = \varphi_6|E(W_3)$.

Then a P_3-isomorphism τ is said to be of *Whitney type i* if τ or τ^{-1} is equivalent to a diamond inflation of φ_i as above for $i = 3, 4, 5, 6$.

Denote by t_z the number of terminal edges incident with z for z in $\{a, b, c, d\}$ or $\{u, v, w, x\}$. For Whitney type P_3-isomorphisms, according to condition (ii) of Diamond Inflation, gives one equation from each pair of corresponding edges of the original Whitney graphs. Then there is a same solution for all four types:

$$\begin{cases} t_u = \frac{1}{2}(t_a + t_b + t_c - t_d) \\ t_v = \frac{1}{2}(t_a + t_b - t_c + t_d) \\ t_w = \frac{1}{2}(t_a - t_b + t_c + t_d) \\ t_x = \frac{1}{2}(-t_a + t_b + t_c + t_d) \quad \text{(except for type 3)} \end{cases} \tag{1}$$

Because we require connected P_3-graphs, in the above four equations we must have $t_z = 0$ or 1 for every $z \in \{a, b, c, d\} \cup \{u, v, w, x\}$. We write $(t_a, t_b, t_c, t_d) \mapsto (t_u, t_v, t_w, t_x)$. If t_a, t_b, t_c, $t_d = 0$ or 1, then we get the corresponding solutions for t_u, t_v, t_w, t_x by (1). For example: (1, 0, 0, 1) $\mapsto$ (0, 1, 1, 0) denotes that $t_a = 1$, $t_b = t_c = 0$ and $t_d = 1$ correspond to solutions $t_u = 0$, $t_v = t_w = 1$ and $t_x = 0$ by (1). So it is easy to check that there are only the following eight cases satisfying $t_z = 0$ or 1 for every $z \in \{a, b, c, d\} \cup \{u, v, w, x\}$:

(i) (0, 0, 0, 0) $\mapsto$ (0, 0, 0, 0).
(ii) (1, 1, 1, 1) $\mapsto$ (1, 1, 1, 1) (except for type 3).
(iii) (1, 1, 0, 0) $\mapsto$ (1, 1, 0, 0) ($ab \mapsto uv$).
(iv) (1, 0, 1, 0) $\mapsto$ (1, 0, 1, 0) ($ac \mapsto uw$).
(v) (1, 0, 0, 1) $\mapsto$ (0, 1, 1, 0) ($ad \mapsto vw$).
(vi) (0, 1, 1, 0) $\mapsto$ (1, 0, 0, 1) ($bc \mapsto ux$) (except for type 3).
(vii) (0, 1, 0, 1) $\mapsto$ (0, 1, 0, 1) ($bd \mapsto vx$) (except for type 3 or 4).
(viii) (0, 0, 1, 1) $\mapsto$ (0, 0, 1, 1) ($cd \mapsto wx$) (except for type $3, 4$ or 5).

If a P_3-isomorphism τ or τ^{-1} is equivalent to a diamond inflation of φ_i as above, and falls into one of the above cases (i) through (viii), then τ is said to be of *special Whitney type* i for $i = 3, 4, 5$ or 6. Thus only in special Whitney type i for $i = 3, 4, 5$ or 6, we can find pairs of nonisomorphic connected graphs with isomorphic connected P_3-graphs if we choose suitable diamond widths.

2.4 Bipartite type

First, we also introduce the definition of bipartite type. Start with a positive integer k and an arbitrary bipartite graph F with at least one edge and with a bipartition (A, B). Let I and I' be different diamond inflations of F, where each edge e is inflated to a diamond of the same width s_e both times, but in producing I each vertex v has t_v terminal edges added, while in producing I' it has t'_v terminal edges added. where

$$t'_v = \begin{cases} t_v - k & \text{if } v \in A \\ t_v + k & \text{if } v \in B \end{cases} \tag{2}$$

Thus, we need $t_v \geq k$ for all $v \in A$. Let φ be the identity edge-isomorphism from F to itself. Clearly φ, I and I' satisfy condition (i) of Diamond Inflation, and condition (ii) is satisfied because each edge of F has the form ab with $a \in A$ and $b \in B$, so that $t'_a + t'_b = (t_a - k) + (t_b + k) = t_a + t_b$. We can

therefore obtain a P_3-isomorphism τ by diamond inflation; τ is in general not induced. We say τ and τ^{-1}, or any equivalent P_3-isomorphisms, are of *bipartite type*.

This case is similar to the above Whitney type. Because we require that the P_3-graphs of I and I' are connected, we must have t_v, $t'_v = 0$ or 1 for every $v \in A \cup B$. Since $k \leq t_v (v \in A)$, we have $k = 0$ or 1. If $k = 0$, then $I \cong I'$. If $k = 1$, then $t_u = 1$ for all $u \in A$ and $t_v = 0$ for all $v \in B$. Otherwise, if there is a vertex $u_0 \in A$ with $t_{u_0} = 0$ or a vertex $v_0 \in B$ with $t_{v_0} = 1$, then $t'_{u_0} = -1$ or $t'_{v_0} = 2$ by (2). Therefore we have a P_3-isomorphism τ_0 from I to I', where $t_u = 1$ and $t'_u = 0$ for all $u \in A$, $t_v = 0$ and $t'_v = 1$ for all $v \in B$, respectively. Then we say that τ_0 and τ_0^{-1}, or any equivalent P_3-isomorphism, are of *special bipartite type*. Therefore, this is the only case to find pairs of nonisomorphic connected graphs which have isomorphic connected P_3-graphs in the bipartite type.

2.5 $TBSD$-related to an induced P_3-isomorphism

In this subsection, we require that there is no isolated vertices in P_3-graphs. Then all P_3-isomorphisms are BSD-related to an induced one. It is clear that if two original graphs G and H are connected with an isomorphic P_3-graph, then $G \cong H$ by the definition of BSD-related. Thus in this type, if we require connected P_3-graphs, then the original graph and its P_3-graph are one to one.

Then from the arguments in above five subsections, we can get the following corollary which is essential to the solution of our problem.

Corollary 2.3 *Let τ be a P_3-isomorphism from G to H, where G and H are nonisomorphic connected graphs with an isomorphic connected P_3-graph. Then τ is one of the following:*

(i) of special Whitney type;

(ii) D-related to a P_3-isomorphism of special Whitney type 3, 4, 5 *or* 6*; or*

(iii) D-related to a P_3-isomorphism of special bipartite type.

3 Main result

Now we can state and show the main result of this paper.

Theorem 3.1 *There is no triple of mutually nonisomorphic connected graphs with an isomorphic connected P_3-graph.*

Proof. Assume, to the contrary, that there exists a triple of mutually nonisomorphic connected graphs G_1, G_2 and G_3 which have an isomorphic connected P_3-graph. Let τ_i be a P_3-isomorphism from G_i to G_{i+1}, then τ_i will be one of three types in Corollary 2.3 for $i = 1, 2$.

Case 1. τ_1 and τ_2 are of the same type.

Subcase 1.1 τ_1 and τ_2 are both of special Whitney type.

Without loss of generality, let $G_1 \cong SW$ and $G_2 \cong C_6$. Since τ_2 is also of special Whitney type, it is clear that $G_3 \cong SW$. Thus $G_1 \cong G_3$, a contradiction.

Subcase 1.2 τ_1 and τ_2 are both of D-related to a P_3-isomorphism of special Whitney type i for $i = 3, 4, 5$ or 6.

Without loss of generality, we assume that $i = 4$. Then τ_1 and τ_2 are D-related to a P_3-isomorphism of special Whitney type 4, and let G_1 and G_2 be diamond inflations of W_4 and $W_4^{'}$, respectively, where $t_a = 1$, $t_b = t_c = 0$, $t_d = 1$, $t_u = 0$, $t_v = t_w = 1$ and $t_x = 0$ (i.e., $(1, 0, 0, 1) \mapsto (0, 1, 1, 0)$). Since τ_1 and τ_2 are of the same type, G_3 is also a diamond inflation of W_4, and $t_a = 1$, $t_b = t_c = 0$, $t_d = 1$ by (1). Hence $G_1 \cong G_3$, also a contradiction.

Subcase 1.3 τ_1 and τ_2 are both of D-related to a P_3-isomorphism of special bipartite type.

This subcase is similar to Subcase 1.2. Denote by F an arbitrary bipartite graph with a bipartition (A, B). Then assume that G_1 and G_2 are different diamond inflations of F, respectively, where $t_u = 1$ for all $u \in A$ and $t_v = 0$ for all $v \in B$ in G_1; $t_u = 0$ for all $u \in A$ and $t_v = 1$ for all $v \in B$ in G_2. Thus we can easily obtain that G_3 is also a diamond inflation of F with $t_u = 1$ for all $u \in A$ and $t_v = 0$ for all $v \in B$ in G_3 by the definition of τ_2. Then $G_1 \cong G_3$, contrary to the assumption.

Case 2. τ_1 and τ_2 are of different types.

By the definition of special Whitney type, we know that it is also a particular case of special Whitney type 3, with the following restrictions: (i) each edge e with diamond width $s_e = 1$ in $K_{1,3}$ and K_3, and (ii) $t_u = 0$ for each vertex u in $K_{1,3}$ and K_3. In fact, special Whitney type is the same as special Whitney type 3 in essence. Then in order to solve Case 2, we only need to distinguish the following two subcases:

Subcase 2.1 τ_1 is of special Whitney type, and τ_2 is D-related to a P_3-isomorphism of special bipartite type.

By the definition of τ_1, G_1 or G_2 is a diamond inflation of $W_3 = K_{1,3}$ or $W_3^{'} = K_3$; and also by τ_2, G_2 and G_3 are different diamond inflations of some bipartite graph, respectively. Thus there is only one possibility: G_2 is a diamond inflation of $K_{1,3}$, where $K_{1,3}$ has a bipartition $A = \{a\}$, $B = \{b, c, d\}$. Then $G_1 \cong C_6$ and $G_2 \cong SW$. It is easy to see that $t_a = t_b = t_c = t_d = 0$ in $K_{1,3}$, a contradiction to the definition of special bipartite type,

where $t_a = 1$ or $t_b = t_c = t_d = 1$.

Subcase 2.2 τ_1 is D-related to a P_3-isomorphism of special Whitney type i for $i = 3, 4, 5$ or 6, and τ_2 is D-related to a P_3-isomorphism of special bipartite type.

For $i = 4, 5$ or 6, if τ_1 is D-related to a P_3-isomorphism of special Whitney type i, then G_1 and G_2 are diamond inflations of W_i and W_i' which have odd cycles. G_2 and G_3 are different diamond inflations of some bipartite graph by the definition of τ_2. Then τ_1 must be D-related to a P_3-isomorphism of special Whitney type 3. By the same argument as in Subcase 2.1, we obtain that G_2 is a diamond inflation of $K_{1,3}$, where $K_{1,3}$ has a bipartition $A = \{a\}$, $B = \{b, c, d\}$. By the definition of special Whitney type 3, τ_1 falls into one of the following four cases: (0, 0, 0, 0) $\mapsto$ (0, 0, 0, 0), (1, 1, 0, 0) $\mapsto$ (1, 1, 0, 0) ($ab \mapsto uv$), (1, 0, 1, 0) $\mapsto$ (1, 0, 1, 0) ($ac \mapsto uw$), or (1, 0, 0, 1) $\mapsto$ (0, 1, 1, 0) ($ad \mapsto vw$). However, by the definition of special bipartite type, there are only two choices: either $t_a = 0$, $t_b = t_c = t_d = 1$, or $t_a = 1$, $t_b = t_c = t_d = 0$. Finally, there does not exist any graph G_2 that has common property of two different types at the same time. So τ_1 and τ_2 must be of the same type, a contradiction. The proof is thus complete. ∎

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